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Rank of Handelman hierarchy for Max-Cut

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ABSTRACT

We consider a hierarchical relaxation, called Handelman hierarchy, for a class of polynomial optimization problems. We prove that the rank of Handelman hierarchy, if applied to a standard quadratic formulation of Max-Cut, is exactly the same as the number of nodes of the underlying graph. Also we give an error bound for Handelman hierarchy, in terms of its level, applied to the Max-Cut formulation.

1. Introduction

Consider the following problem where $p(x), g_1(x), \ldots, g_m(x) \in \mathbb{R}[x]$ are real polynomials in n variables.

$$p^{\min} = \min \quad p(x) \qquad \text{s.t.} \quad x \in K := \left\{x \in \mathbb{R}^n | g_1(x) \geq 0, \ldots, g_m(x) \geq 0\right\}. \tag{1}$$

The *polynomial optimization problem* (1) is NP-hard. Indeed, if $p(x)$ is quadratic and $g_i(x)$'s are linear, problem (1) contains the NP-hard Max-Cut problem; see relation (8) in Section 3.

Problem (1) can be rewritten as sup ρ s.t. $p(x) - \rho \geq 0$ over K, i.e.,

$$p^{\min} = \sup \rho \quad \text{s.t. } p(x) - \rho \in \mathcal{P}(K), \tag{2}$$

where $\mathcal{P}(K)$ denotes the set of real polynomials that are nonnegative on K. However, the intractability of (1) implies that the problem of testing membership of an arbitrary polynomial in $\mathcal{P}(K)$ is already an intractable problem. Several relaxation ideas for (2) have been proposed in the literature. They replace $\mathcal{P}(K)$ with its subsets whose membership problems are tractable. They rely on various characterizations, known as *Positivstellensatz*, of polynomials that are positive over the semialgebraic set K [8,15,14].

In this paper, we consider a class of polynomial optimization problems in which K is restricted to be a compact and solid polyhedral set. In this case, as observed by [7], a Positivstellensatz by Handelman [8] provides a hierarchical relaxation of (1) that converges to an exact optimum. In Section 2, we review Handelman hierarchy and its polynomiality of the relaxation of a fixed level. Section 3 is devoted to establishing the rank of Handelman hierarchy for Max-Cut. In Section 4, we derive an upper bound on the error of Handelman hierarchy with respect to the exact optimum value of Max-Cut. Section 5 observes the duality between Handelman hierarchy and RLT and its implications on the rank of RLT with respect to Max-Cut.

2. Handelman hierarchy

Our notation is mainly adopted from [12]. Given a set of polynomials $g_1(x), \ldots, g_m(x) \in \mathbb{R}[x]$ defining K and $\beta \in \mathbb{N}^m$, we denote $g^\beta := g_1^{\beta_1} \cdots g_m^{\beta_m}$. For notational convenience, for any g_i, we also denote $g_i^0 := 1$. We assume the set $\mathbb{N}^m$ is graded lexicographically ordered and the degree of $\alpha \in \mathbb{N}^m$ denoted by $|\alpha|$ is the sum of entries, i.e., $|\alpha| = \sum_{i=1}^m \alpha_i$. Let $\mathbb{N}_t^m$ be the set of m-dimensional vectors whose degree is at most t. It is well known that $|\mathbb{N}_t^m| = \binom{m+t}{t}$. We use $\mathbb{R}^{\mathbb{N}_t^m}$ to denote the set of $\binom{m+t}{t}$-dimensional real vectors $y := (y_\alpha : \alpha \in \mathbb{N}_t^m)$.

Given a polynomial $f(x) \in \mathbb{R}[x]$, $\deg(f(x))$ denotes the degree of $f(x)$. Also we adopt a little abusive but convenient notation that $f \in \mathbb{R}^{\mathbb{N}_t^n}$ denotes the vector whose α-th entry, f_α, is the coefficient of x^α in the polynomial $f(x)$. We will use $\mathrm{vec}(f(x))$ instead of f especially when $f(x)$ is given in a specific form. Given a set of polynomials S, $\mathrm{vec}(S) := \{\mathrm{vec}(f(x)) \mid f(x) \in S\}$.

Consider a set of polynomials

$$H(g) := \left\{ \sum_{\beta \in \mathbb{N}^m} c_\beta g^\beta | c_\beta \in \mathbb{R}_+ \right\},$$

known as the *preprime* generated by $g_1(x), \ldots, g_m(x)$. It is easy to see that each polynomial in $H(g)$ is nonnegative on K and hence

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$H(g) \subseteq \mathscr{P}(K)$. Define

$$p^{\text{han}} = \sup \ \rho \quad \text{s.t. } p(x) - \rho \in H(g). \tag{3}$$

Then (3) is a relaxation of (2) in the sense that $p^{\text{han}} \le p^{\min}$. However, the membership of an arbitrary real polynomial to $H(g)$ is not yet an easy problem. The Handelman hierarchy is obtained by further substituting $H_t(g)$ for $H(g)$ [7]:

$$p_t^{\text{han}} = \sup \ \rho \quad \text{s.t. } p(x) - \rho \in H_t(g), \tag{4}$$

where

$$H_t(g) := \left\{ \sum_{\beta \in \mathbb{N}^m} c_\beta g^\beta \,|\, \deg(g^\beta) \le t, c_\beta \in \mathbb{R}_+ \right\}. \tag{5}$$

It is easy to see that, for every $t \ge \deg(p(x))$, we have $p_t^{\text{han}} \le p^{\min}$. Also, since $H_t(g) \subseteq H_{t+1}(g)$, we have $p_t^{\text{han}} \le p_{t+1}^{\text{han}}$.

It is well-known from the duality theorem for linear programming that if $p(x)$ is linear, K is a polyhedron, and $p(x)$ attains an optimal solution over K, then $p(x) - p^{\min}$ is a conic combination of $g_i(x)$'s. Thus, in this case, $p_1^{\text{han}} = p^{\min}$:

$$p_1^{\text{han}} = \sup \ \rho \quad \text{s.t. } p(x) - \rho \in H_1(g).$$

Of course, this is not true in general as $H(g)$ is a proper subset of $\mathscr{P}(K)$. However, when K is a compact and solid polyhedron, p_t^{han} is guaranteed to converge to $p^{\min}$.

Theorem 2.1 (*Handelman's Positivstellensatz [8]*). *Suppose K is a compact polyhedral set which is solid, i.e., it has an interior point. If $p(x)$ is positive on K, then $p(x) \in H(g)$.*

Now we consider the computational complexity of problem (4) for a fixed t. Suppose $\deg(g^\beta) \le t$. Then we can write

$$g^\beta(x) = \sum_{\alpha \in \mathbb{N}_t^n} a_{\alpha,\beta} x^\alpha.$$

Define the matrix A_t whose (α, β)-th element is $a_{\alpha,\beta}$ for $\alpha \in \mathbb{N}_t^n$ and $\beta \in \mathbb{N}_t^m$ with $\deg(g^\beta) \le t$. Let z stand for the vector of coefficients c_β defined in (5) and $e_0 \in \mathbb{R}^{\mathbb{N}_t^n}$ be the vector whose α-th entry is 1 if $\alpha = 0$, and 0, otherwise. Then problem (4) is equivalent to the following LP problem which computes z that maximizes ρ:

$$\begin{aligned} p_t^{\text{han}} = \sup \quad & \rho \\ \text{s.t.} \quad & \rho e_0 + A_t z = p \\ & z \ge 0. \end{aligned} \tag{6}$$

The number of variables is less than or equal to $\binom{m+t}{t}$ and the number of constraints is equal to $\binom{n+t}{t}$. Thus for a fixed t, p_t^{han} can be obtained by an LP whose dimension is polynomial in m and n. (However, the dimension increases exponentially in t.)

We call the *rank* of the Handelman hierarchy with respect to a problem the minimum t such that $p_t^{\text{han}} = p^{\min}$ for every instance of the problem. There has been a significant literature on the rank of various hierarchies for combinatorial optimization problems [13,2,18,6,1,4,5,9]. In the following section, we show that the rank of the Handelman hierarchy, if applied to a standard quadratic formulation of Max-Cut, is exactly the number of nodes of the underlying graph.

3. Application to Max-Cut

The maximum cut problem, or Max-Cut, is the problem of finding a partition $(V_1; V_2)$ of the nodes $V = \{1, \ldots, n\}$ of a given graph $G = (V, E)$ that maximizes the number of edges between V_1 and V_2.

Max-Cut is often formulated as a binary quadratic program. Let $A_G \in \{0, 1\}^{n \times n}$ be the adjacency matrix of G: its (i, j)-th entry is 1 if $ij \in E$, and 0, otherwise. Also, define $x \in \{0, 1\}^n$ as follows:

$$x_i = \begin{cases} 1, & \text{if } i \in V_1, \\ 0, & \text{if } i \in V_2. \end{cases}$$

Then it is easy to see that nodes i and j, if they belong to distinct sets and $(i, j) \in E$, contribute to the objective a unit which can be written as $(A_G)_{ij} x_i (1 - x_j) + (A_G)_{ji} x_j (1 - x_i)$. Thus the following 0–1 quadratic program formulates Max-Cut:

$$\max_{x \in \{0,1\}^n} x^T A_G (e - x), \tag{7}$$

where $e \in \mathbb{R}^n$ is the vector of all ones. Now, in (7), we relax the binary restriction $x \in \{0, 1\}^n$ to membership in the hypercube leading to

$$\max_{x \in [0,1]^n} x^T A_G (e - x). \tag{8}$$

Since every diagonal element of A_G is 0, the objective $x^T A_G (e - x)$ is linear with respect to each variable x_i. This implies that there is an optimal solution of (8) in which each x_i is either 0 or 1. Thus (8) is an exact formulation of Max-Cut. Since the feasible solution set is a hypercube, Handelman hierarchy applied to (8) will converge to the optimal value.

Now we show that its rank is exactly n. To do so, we use the minimization version of (8).

$$\begin{aligned} p^{\min} = \min \quad & -x^T A_G (e - x) \\ \text{s.t.} \quad & x \in Q := \{x \in \mathbb{R}^n | x_i \ge 0, 1 - x_i \ge 0, \\ & \quad i = 1, \ldots, n\}. \end{aligned} \tag{9}$$

Let $g_i = x_i$, and $g_{n+i} = 1 - x_i$ for $i = 1, \ldots, n$.

An upper bound on the rank can be derived from the following theorem.

Theorem 3.1 (*De Klerk and Laurent [7]*). *Let $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and $p(x) = x^T A x + b^T x$. Consider the problem of minimizing $p(x)$ over the hypercube Q. Then for any integer $k \ge 1$, there is $t \le \max(kn, 2)$ such that*

$$p(x) - p^{\min} + \frac{1}{k} \sum_{i=1}^n \max\{A_{ii}, 0\} \in H_t(g).$$

In other words, $p^{\min} - p_t^{\text{han}} \le \frac{1}{k} \sum_{i=1}^n \max\{A_{ii}, 0\}$ when $t = \max(kn, 2)$ for any integer $k \ge 1$.

Corollary 3.2. *Suppose $n \ge 2$. Then the rank of Handelman hierarchy, if applied to (9), is not greater than n.*

Proof. The corollary is immediate from Theorem 3.1 and the fact that every diagonal element of A_G is 0. □

Now we establish the lower bound, namely, that the rank is at least n. To do so, the following lemma is useful.

Lemma 3.3. *If there is an instance of a problem such that $p(x) - p^{\min} \notin H_{t-1}(g)$ in (4), then the rank of the Handelman hierarchy is not less than t.*

Proof. Suppose there is an instance such that $p(x) - p^{\min} \notin H_{t-1}(g)$. Since $\text{vec}(H_{t-1}(g))$ is a closed convex set, there is a hyperplane strictly separating $\text{vec}(p(x) - p^{\min})$ from $\text{vec}(H_{t-1}(g))$. Hence, there is an $\epsilon > 0$ such that $p(x) - p^{\min} + \epsilon \notin H_{t-1}(g)$. Also, if $p(x) - \sigma \notin H_{t-1}(g)$, then $p(x) - \rho \notin H_{t-1}(g)$ for every $\rho > \sigma$. Therefore, we have $p_{t-1}^{\text{han}} \le p^{\min} - \epsilon < p^{\min}$. This implies that t is less than or equal to the rank. □

From Lemma 3.3, it suffices to construct an instance for which $p(x) - p^{\min} \notin H_{n-1}(g)$. Consider a Max-Cut whose underlying graph is the complete graph K_n with $n = 2k+1$ for some integer k. Its min-version optimal value is $p^{\min} = -(k^2+k)$. Thus, we need to show that

$$p_n(x) := -x^T A_{K_n}(e-x) + k(k+1)$$
$$= -2k\sum_{i=1}^{n} x_i + 2\sum_{ij\in E} x_i x_j + k(k+1) \notin H_{n-1}(g). \tag{10}$$

Theorem 3.4. $p_n(x) \notin H_{n-1}(g)$.

Proof. Condition (10) is equivalent to the existence of a hyperplane $a^T w = 0$ strictly separating p_n from $\operatorname{vec}(H_{n-1}(g))$, i.e. $a^T p_n < 0$ and $a^T g^\beta \geq 0$ for every $\beta \in \mathbb{N}^{2n}_{n-1}$. Denote by $\Upsilon_i \in \mathbb{R}^{\mathbb{N}^n_{n-1}}$ the vector whose α-th entry is 1 if $|\alpha| = i$, and 0, otherwise. We will show if we set

$$a_n = \binom{2k}{k}\Upsilon_0 + \binom{2k-1}{k-1}\Upsilon_1 + \cdots + \binom{2k-i}{k-i}\Upsilon_i + \cdots + \binom{k}{0}\Upsilon_k,$$

then $a_n^T w = 0$ is a hyperplane strictly separating p_n from vec $(H_{n-1}(g))$.

First, we will show $a_n^T p_n < 0$. Suppose $k = 1$. Then, since $a_3 = 2\Upsilon_0 + \Upsilon_1$, we get

$$a_3^T p_3 = 2\times 2 - 2\times 3 = -2 < 0.$$

If $k \geq 2$, we have

$$\begin{aligned}
a_n^T p_n &= \binom{2k}{k}k(k+1) - \binom{2k-1}{k-1}(2k+1)\cdot 2k + \binom{2k-2}{k-2}(2k+1)\cdot 2k \\
&= \frac{2k(2k-1)}{k(k-1)}\binom{2k-2}{k-2}k(k+1) - \frac{2k-1}{k-1}\binom{2k-2}{k-2}(2k+1)\cdot 2k + \binom{2k-2}{k-2}(2k+1)\cdot 2k \\
&= \frac{2k}{k-1}\binom{2k-2}{k-2}[(2k-1)(k+1) - (2k-1)(2k+1) + (k-1)(2k+1)] \\
&= -\frac{2k}{k-1}\binom{2k-2}{k-2} < 0.
\end{aligned}$$

It remains to show that $a_n^T g^\beta \geq 0$ for every $\beta \in \mathbb{N}^{2n}_{n-1}$. Write $g^\beta(x) = x^\alpha(e-x)^\gamma$. Note that for any real polynomial $h(x)$, $\Upsilon_i^T h$ is the sum of coefficients of monomials in $h(x)$ whose degree is i. Hence $\Upsilon_i^T h = \Upsilon_i^T \operatorname{vec}(h(x_1, x_1, \ldots, x_1))$. Then, for every i, $\Upsilon_i^T g^\beta$ is 0 if $i < |\alpha|$. If, on the other hand, $i \geq |\alpha|$, we have

$$\begin{aligned}
\Upsilon_i^T g^\beta &= \Upsilon_i^T \operatorname{vec}(x^\alpha(e-x)^\gamma) \\
&= \Upsilon_i^T \operatorname{vec}(x_1^{|\alpha|}(1-x_1)^{|\gamma|}) \\
&= \Upsilon_{i-|\alpha|}^T \operatorname{vec}((1-x_1)^{|\gamma|}) \\
&= (-1)^{i-|\alpha|}\binom{|\gamma|}{i-|\alpha|}.
\end{aligned} \tag{11}$$

From (11), for each β, $\Upsilon_i^T g^\beta$ is determined only by $|\alpha|$ and $|\gamma|$. Relying on the observation, we can show $a_n^T g^\beta \geq 0$ when $|\beta| = n-1$. Suppose $k+1 \leq |\alpha| \leq n-1$. Then $\Upsilon_i^T g^\beta = 0$ for $i = 0, \ldots, k$ and, hence, $a_n^T g^\beta = \sum_{i=0}^{k}\binom{k+i}{i}\Upsilon_{k-i}^T g^\beta = 0$. Suppose, on the other hand, $0 \leq |\alpha| \leq k$ and let $j = k - |\alpha|$. Then we get $|\gamma| = k+j$ and hence

$$\begin{aligned}
a_n^T g^\beta &= \sum_{i=0}^{j}\binom{k+i}{i}\Upsilon_{k-i}^T g^\beta \\
&= \sum_{i=0}^{j}\binom{k+i}{i}(-1)^{k-i-|\alpha|}\binom{|\gamma|}{k-i-|\alpha|} \\
&= \sum_{i=0}^{j}\binom{k+i}{i}(-1)^{j-i}\binom{k+j}{j-i} \\
&= \sum_{i=0}^{j}(-1)^{j-i}\frac{(k+i)!}{i!k!}\frac{(k+j)!}{(j-i)!(k+i)!} \\
&= \sum_{i=0}^{j}(-1)^{j-i}\frac{(k+j)!}{k!j!}\frac{j!}{i!(j-i)!} \\
&= \binom{k+j}{j}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{j-i} \\
&= \binom{k+j}{j}(1-1)^j \geq 0.
\end{aligned}$$

Finally, we prove if $a_n^T g^\beta \geq 0$ for every β with $|\beta| = l$, then it also holds for every β with $|\beta| = l-1$. To do so, it suffices to establish the following for $i = 0, \ldots, k$:

$$\Upsilon_i^T \operatorname{vec}(x_1^{|\alpha|}(1-x_1)^{|\gamma|}) = \Upsilon_i^T \operatorname{vec}(x_1^{|\alpha|}(1-x_1)^{|\gamma|+1}) + \Upsilon_i^T \operatorname{vec}(x_1^{|\alpha|+1}(1-x_1)^{|\gamma|}). \tag{12}$$

But, if $i < |\alpha|$, (12) holds since both sides are 0. Suppose $i \geq |\alpha|$. Relying on the identity $\binom{a+1}{b} - \binom{a}{b-1} = \binom{a}{b}$ for every nonnegative integers a, b, we have

$$\begin{aligned}
\text{Right-hand-side of (12)} &= (-1)^{i-|\alpha|}\binom{|\gamma|+1}{i-|\alpha|} + (-1)^{i-|\alpha|-1}\binom{|\gamma|}{i-|\alpha|-1} \\
&= (-1)^{i-|\alpha|}\binom{|\gamma|}{i-|\alpha|} \\
&= \text{Left-hand-side of (12)}.
\end{aligned}$$

Thus we have completed the proof. □

Proposition 3.5. *The rank of Handelman hierarchy, applied to formulation (8) of Max-Cut, is equal to the number of nodes of the underlying graph.*

We can observe that when G is bipartite, the rank is 2.

Proposition 3.6. $p_2^{\text{han}} = |E|$.

Proof. Define $p(x) := x^T A_G(e-x) = \sum_i \delta_i x_i - 2\sum_{ij\in E} x_i x_j$, where δ_i is the number of edges incident to the node i in G. Then

$$\begin{aligned}
|E| - p(x) &= |E| - \sum_i \delta_i x_i + 2\sum_{ij\in E} x_i x_j \\
&= \sum_{ij\in E}[(1-x_i)(1-x_j) + x_i x_j] \in H_2(g).
\end{aligned}$$

Hence $p_2^{\text{han}} \leq |E|$.

Now, we will show that if $\lambda - p(x) \in H_2(g)$, then $\lambda \geq |E|$. Suppose that $\lambda - p(x) \in H_2(g)$, that is, $\lambda - \sum_i \delta_i x_i + 2\sum_{ij\in E} x_i x_j$ has a decomposition of the form:

$$a_0 + \sum_i a_i x_i + \sum_i b_i(1-x_i) + \sum_i c_i x_i^2 + \sum_i d_i(1-x_i)^2 + \sum_{i<j} e_{ij}x_i x_j + \sum_{i<j} f_{ij}(1-x_i)(1-x_j) + \sum_{i,j} g_{ij}x_i(1-x_j)$$

for some nonnegative scalars $a_i, b_i, c_i, d_i, e_{ij}, f_{ij}, g_{ij}$.

Looking at the constant coefficient we get

$$\lambda = a_0 + \sum_i b_i + \sum_i d_i + \sum_{i<j} f_{ij}. \tag{13}$$

Looking at the coefficient of x_i we get

$$-\delta_i = a_i - b_i - 2d_i - \sum_{j|j\neq i} f_{ij} + \sum_j g_{ij},$$

which implies

$$b_i + 2d_i + \sum_{j|j\neq i} f_{ij} = \delta_i + a_i + \sum_j g_{ij}. \tag{14}$$

Summing up over i in (14), we obtain

$$\sum_i b_i + 2\sum_i d_i + 2\sum_{i<j} f_{ij} = \sum_i \delta_i + \sum_i a_i + \sum_{i,j} g_{ij} \geq 2|E|, \tag{15}$$

where we use the facts that $\sum_i \delta_i = 2|E|$ and $a_i, g_{ij} \geq 0$. Using (13), we obtain:

$$2\lambda = \underbrace{\left(2a_0 + \sum_i b_i\right)}_{\geq 0} + \underbrace{\left(\sum_i b_i + 2\sum_i d_i + 2\sum_{i<j} f_{ij}\right)}_{\geq 2|E| \text{ by } (15)} \geq 2|E|.$$

This concludes the proof. □

Since OPT $= |E|$ when G is bipartite, the above proposition shows the rank is not greater than 2. But, the minimum value of t for which Handelman hierarchy is defined is the degree of the objective function. Therefore, it follows that the rank is 2 when G is bipartite.

4. Error of Handelman hierarchy for Max-Cut

The error bound for Handelman hierarchy from Theorem 3.1 only applies to the case when t is a multiple of n. As the rank of Handelman hierarchy is n for (8), we need to explore an error bound for the case when $t < n$.

The idea is similar to the one used in [7]. First, we approximate $p(x)$ with a linear combination of polynomials from $H_t(g)$. Second, we show that $p(x) - p^{\min}+$ some constant (depending on t) $\epsilon(t)$ is a polynomial in $H_t(g)$ using the relation between $p(x)$ and the approximation. Then it follows that $\epsilon(t)$ is an upper bound on the error $p^{\min} - p_t^{\text{han}}$.

In this paper, to derive an error bound for $t < n$ from $H_t(g)$, we will choose a set of polynomials that are different from those used in [7]. Given $S \subseteq V$ s.t. $|S| = t$ and $\alpha \in \{0,1\}^S$, define

$$P_{S,\alpha}(x) = \prod_{i\in S} x_i^{\alpha_i}(1-x_i)^{1-\alpha_i}.$$

Notice that $P_{S,\alpha}(x) \in H_t(g)$. We approximate $f(x)$, a continuous function on $[0,1]^n$, as a linear combination of $P_{S,\alpha}(x)$'s:

$$B_t(f(x)) = \sum_{S\subseteq V,|S|=t} \sum_{\alpha\in\{0,1\}^S} f(\tilde{\alpha})P_{S,\alpha}(x), \tag{16}$$

where $\tilde{\alpha} \in \{0,1\}^n$ is a vector whose i-th entry is α_i if $i \in S$, and 0, otherwise. Then, for each t, it is easy to check that B_t is a linear operator: $B_t(cf(x) + \mathrm{d}g(x)) = cB_t(f(x)) + \mathrm{d}B_t(g(x))$.

Applying approximation (16) to the Max-Cut objective, $f(x) = -\sum_{i\in V}\delta_i x_i + 2\sum_{ij\in E} x_i x_j$, we get

$$B_t(f(x)) = -\sum_{i\in V}\delta_i B_t(x_i) + 2\sum_{ij\in E} B_t(x_i x_j). \tag{17}$$

Note that

$$B_t(x_i) = \sum_{S\subseteq V,|S|=t}\sum_{\alpha\in\{0,1\}^S} \tilde{\alpha}_i P_{S,\alpha}(x) = \sum_{S\subseteq V,|S|=t,i\in S}\sum_{\alpha\in\{0,1\}^S,\alpha_i=1} P_{S,\alpha}(x) = \sum_{S\subseteq V,|S|=t,i\in S} x_i = \binom{n-1}{t-1}x_i.$$

Similarly, we get

$$B_t(x_i x_j) = \sum_{S\subseteq V,|S|=t}\sum_{\alpha\in\{0,1\}^S} \tilde{\alpha}_i\tilde{\alpha}_j P_{S,\alpha}(x) = \sum_{S\subseteq V,|S|=t,i,j\in S}\sum_{\alpha\in\{0,1\}^S,\alpha_i=\alpha_j=1} P_{S,\alpha}(x) = \sum_{S\subseteq V,|S|=t,i,j\in S} x_i x_j = \binom{n-2}{t-2}x_i x_j.$$

Substituting these two for those of (17), we have

$$B_t(f(x)) = -\sum_{i\in V}\delta_i\binom{n-1}{t-1}x_i + 2\sum_{ij\in E}\binom{n-2}{t-2}x_i x_j. \tag{18}$$

Add $2\sum_{ij\in E}\frac{n-t}{t-1}\binom{n-2}{t-2}x_i x_j$ to both sides of (18) and then divide them by $\binom{n-1}{t-1}$ to get

$$B_t(f(x))/\binom{n-1}{t-1} + 2\sum_{ij\in E}\frac{n-t}{n-1}x_i x_j = f(x). \tag{19}$$

Again, add $\frac{n}{t}$OPT to both sides of (19) to get

$$\begin{aligned} f(x) + \frac{n}{t}\mathrm{OPT} &= \binom{n}{t}\mathrm{OPT}/\binom{n-1}{t-1} + B_t(f(x))/\binom{n-1}{t-1} + 2\sum_{ij\in E}\frac{n-t}{n-1}x_i x_j \\ &= \sum_{S\subseteq V,|S|=t}\sum_{\alpha\in\{0,1\}^S} P_{S,\alpha}(x)\mathrm{OPT}/\binom{n-1}{t-1} + \sum_{S\subseteq V,|S|=t}\sum_{\alpha\in\{0,1\}^S} P_{S,\alpha}(x)f(\tilde{\alpha})/\binom{n-1}{t-1} + 2\sum_{ij\in E}\frac{n-t}{n-1}x_i x_j \\ &= \sum_{S\subseteq V,|S|=t}\sum_{\alpha\in\{0,1\}^S}(\mathrm{OPT}+f(\tilde{\alpha}))P_{S,\alpha}(x)/\binom{n-1}{t-1} + 2\sum_{ij\in E}\frac{n-t}{n-1}x_i x_j. \end{aligned}$$

The second equality follows from the fact that $\sum_{S\subseteq V,|S|=t}\sum_{\alpha\in\{0,1\}^S} P_{S,\alpha}(x) = \binom{n}{t}$ since $\sum_{\alpha\in\{0,1\}^S} P_{S,\alpha}(x) = 1$. As $P_{S,\alpha}(x), x_i x_j \in H_t(g)$ and the coefficients are nonnegative, since $f(\tilde{\alpha}) = -|(T,\overline{T})|$ where

$T = \{i \in V | \tilde{\alpha}_i = 1\}$, we obtain that $f(x) + \frac{n}{t}\mathrm{OPT} \in H_t(g)$. Thus we get the error bound

$$p_t^{\mathrm{han}} \le \frac{n}{t}\mathrm{OPT}, \quad \text{for all } t \le n. \tag{20}$$

Notice that the error bound (20) is consistent with the rank n proved in Section 3. This bound, however, is meaningful only when $t \ge n/2$. Indeed, the Handelman bound is equal to $|E|$ by Proposition 3.6 while OPT of Max-Cut is at least $|E|/2$. Hence the Handelman hierarchy always provides a bound at most 2OPT.

5. Handelman hierarchy and RLT

In this section, we show that the Handelman hierarchy is dual to the Reformulation Linearization Technique(RLT). RLT is originally proposed by Sherali and Adams [16] for the computation of the convex hull P_I of the 0–1 lattice points from a polyhedron $P = \{x \in \{0,1\}^n | g_l(x) \ge 0,\ l = 1, \dots, m\}$.

It computes a set of valid inequalities of P_I by using the bound inequalities, $0 \le x_i \le 1$. Given a pair of subsets $I, J \subseteq \{1, 2, \dots, n\}$, define

$$f(I,J)(x) = \prod_{i \in I} x_i \cdot \prod_{j \in J} (1 - x_j).$$

Multiply each constraint by $f(I,J)(x)$ to get the following set of valid inequalities:

$$\mathcal{G}_{I,J} := \{g_l(x) \cdot f(I,J)(x) \ge 0 | l = 1, \dots, m\}.$$

For each fixed t, RLT constructs the valid inequalities $\mathcal{G}_{I,J}$ for all pairs I and J such that $|I \cup J| = t$ and $I \cap J = \emptyset$. Then, linearize each valid inequality first by replacing x_i^2 with x_i, and then by substituting y_α for each monomial x^α.

Denote by P_t the projection of the intersection of the set $\{y : y_\emptyset = 1\}$ and the polyhedron defined by the linearized inequalities to the original space. Then, it can be shown that $P_1 \supseteq P_2 \supseteq \cdots \supseteq P_{n-1} \supseteq P_n = P_I$. RLT uses the relaxation of the problem in which P_I is relaxed to P_t for a fixed $t \le n$.

Charikar et al. [3] explored the computational efficiency of RLT, when applied to the IP-formulation of Max-Cut based on the triangle inequalities among edge variables. They showed that for every $\epsilon > 0$ there exists $\gamma > 0$ such that the integrality gap of the above relaxation for $t = \mathcal{O}(n^\gamma)$ is at least $2 - \epsilon$.

Sherali and Tuncbilek [17] extended RLT to the polynomial optimization problems in an analogous manner, using the variable bounds $0 \le l_i \le x_i \le u_i < \infty$. Namely, construct

$$f(I,J)(x) = \prod_{i \in I} (x_i - l_i) \cdot \prod_{j \in J} (u_j - x_j), \quad \forall I, J \text{ s.t. } |I \cup J| = t.$$

Then linearize each valid inequality by substituting y_α for each monomial x^α.

Now we consider the following variation RLT′ of RLT. Namely, for our semialgebraic set $K = \{x \in \mathbb{R}^n \mid g_i(x) \ge 0,\ i = 1, \dots, m\}$, we use the product of the powers of g_i's. For a fixed t we consider the valid inequalities

$$g^\beta(x) := \prod_{i=1}^{n} g_i(x)^{\beta_i} \ge 0, \quad \forall \beta \in \mathbb{N}^m \text{ s.t. } \deg(g^\beta) \le t$$

and linearize them by substituting y_α for x^α. Then RLT′ for (1) becomes the following LP

$$\begin{aligned} p_t^{\mathrm{han}} = \min \quad & p^T y \\ \text{s.t.} \quad & y_0 = 1 \\ & A_t^T y \ge 0, \end{aligned} \tag{21}$$

where A_t is the same matrix as in (6). It is easy to see that (21) is dual to (6). Thus for each t, Handelman hierarchy is dual to RLT′.

Remark 5.1. After the submission of the first version of this paper, we found that the duality between two hierarchies had already been observed in [10].

From the duality between two hierarchies, we have the following corollary to Proposition 3.5.

Corollary 5.2. *The rank of* RLT′, *applied to the quadratic formulation (8) of Max-Cut is also equal to the number of nodes of the underlying graph.*

Remark 5.3. The duality between the two hierarchies immediately transfers any error bound of one hierarchy to the other one for polynomial optimization on compact and solid polyhedra. See, e.g., Theorem 1.4 from [7].

Remark 5.4. The duality also provides a nonconstructive proof of Theorem 3.4. For Max-Cut, the original RLT by Sherali and Adams [16] is at least as powerful as RLT′. Indeed, the original RLT exploits the additional conditions $x_i^2 = x_i$ for $i = 1, \dots, n$ that are represented by the linear constraints $y_{\alpha+e_i} = y_{\alpha+2e_i}$ for all $\alpha \in \mathbb{N}^n_{t-2}$ and $i = 1, \dots, n$ where $e_i \in \mathbb{R}^n$ is the unit vector whose i-th entry is 1. Besides these linear constraints, RLT′ and the original RLT are identical. Hence the rank of the original RLT is a lower bound on the rank of RLT′. Laurent [11] showed that the rank of the original RLT for Max-Cut is n. Combined with the duality, this implies that the rank of Handelman hierarchy for Max-Cut is at least n.

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