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Unification of lower-bound analyses of the lift-and-project rank of combinatorial optimization polyhedra☆

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Abstract

We present a unifying framework to establish a lower bound on the number of semidefinite-programming-based lift-and-project iterations (rank) for computing the convex hull of the feasible solutions of various combinatorial optimization problems. This framework is based on the maps which are commutative with the lift-and-project operators. Some special commutative maps were originally observed by Lovász and Schrijver and have been used usually implicitly in the previous lower-bound analyses. In this paper, we formalize the lift-and-project commutative maps and propose a general framework for lower-bound analysis, in which we can recapture many of the previous lower-bound results on the lift-and-project ranks.

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1. Introduction

An important subject in both theory and practice of combinatorial optimization involves “computing” the convex hull of the integer points lying in a simply described (usually by facets) polytope. Here, “computing” means generating the facets of the convex hull of integer points either explicitly or implicitly.

For many hard combinatorial optimization problems, we lower our goals from computing all the facets of the convex hull to computing a partial, but still useful subset of the facets of the convex hull. Depending on the approach taken, there are many ways of measuring how complicated a facet of the convex hull is. A traditional theoretical approach is to apply Gomory–Chvátal closures to the original polytope and count the number of major iterations needed to derive a particular facet or all facets of the convex hull. The resulting measure, called *Gomory–Chvátal rank*, has been studied, among others, in [7,8,12].

A less mainstream approach to computing the convex hull is the lift-and-project methods. Such methods have been proposed by Balas [3], Lovász and Schrijver [26], and Sherali and Adams [27]. Sherali and Adams [28] extended their

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lift-and-project methods to more general nonconvex problems (also see the references therein for many applications). Closely related Lovász–Schrijver procedures were generalized to compute the convex hull of any compact set in [18]. Ref. [18] contains the first convergence proof of such a method in that generality. Later, Lasserre [19,20] proposed similar procedures in a less general setting and used results from real-algebraic geometry to establish convergence. While the convergence of lift-and-project methods for 0-1 optimization problems was well understood, convergence proofs for lift-and-project methods on more general nonconvex optimization problems involve different techniques. The focus of the current paper is Lovász–Schrijver lift-and-project methods for 0-1 combinatorial optimization problems, especially the method involving positive semidefiniteness constraints. However, we hope that our approach can be generalized to deal with Sherali–Adams procedures, Lasserre-type methods and the more recently proposed methods of Bienstock and Zuckerberg [6]; also see the analysis in [5].

Cook and Dash [10] were the first to make an explicit connection between the tools for lower-bound proving techniques for the Gomory–Chvátal rank and those for the lift-and-project ranks. Here, we slightly generalize their approach and streamline a proof technique for lower-bound analysis. A main feature of the analysis is that the positive semidefiniteness of a certain matrix in a lifted relaxation is established inductively by a simple convexity argument on positive semidefiniteness preserving linear maps (therefore avoiding the need to work out algebraically, the eigenspaces, eigenvalues, etc. of the matrices of arbitrary size). More specifically, let $L_p \in \mathbb{R}^{n\times m}$, $n \geqslant m$. Then the linear transformation $L_p \cdot L_p^{\mathrm{T}}$ maps any $m \times m$ symmetric positive semidefinite matrix to an $n \times n$ symmetric positive semidefinite matrix. Therefore, the linear map $\sum_p \lambda_p (L_p \cdot L_p^{\mathrm{T}})$ also preserves symmetry and positive semidefiniteness (provided $\lambda_p \geqslant 0$, $\forall p$). Taking convex combinations preserves some other critical properties in addition.

The next section contains definitions and the basic properties of the lift-and-project methods that are studied in this paper. In Section 3 we describe the skeleton of a generic proof via the unified approach. To do so, we also introduce the notions of commutativity of linear maps and the lift-and-project operators. Section 4 is made up from typical elementary examples of the unifying proof technique.

2. The lift-and-project methods

In this section, we review the definitions and some of the previously established basic properties of the lift-and-project procedures. For details and further related results, see [26–28,4,23,10,16,24,13].

Let P be any convex subset of the d-dimensional hypercube $[0, 1]^d$. P_{I} denotes the integral hull of P, namely the convex hull of 0-1 vectors of P. The lift-and-project methods are general procedures which take P as input and deliver P_{I} as output. In doing so, it is sometimes convenient to homogenize P to a cone K in $\mathbb{R}^{d+1}$ by introducing an additional coordinate which will be referred to as the 0th coordinate.

$$K := \left\{ \lambda \begin{pmatrix} 1 \\ x \end{pmatrix} : x \in P, \lambda \geqslant 0 \right\} \quad \text{or} \quad P = \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} \in K \right\}. \tag{1}$$

Accordingly, K_{I} is the homogenized cone of P_{I}. See Fig. 1. It is clear that K is contained in $Q \subseteq \mathbb{R}^{d+1}$, the homogenization of $[0, 1]^d$. The cone Q has a very simple polyhedral structure. Denote $H_i(0) := \{x \in \mathbb{R}^{d+1} : x_i = 0\}$ and $H_i(1) := \{x \in \mathbb{R}^{d+1} : x_i = x_0\}$. Similarly, for $J \subseteq \{1, 2, \ldots, d\}$, write $H_J(0) := \{x \in \mathbb{R}^{d+1} : x_i = 0,\ i \in J\}$ and $H_J(1) := \{x \in \mathbb{R}^{d+1} : x_i = x_0,\ i \in J\}$. Then, for each $(d + 1 - k)$-dimensional face of Q, there is a set $J \subseteq \{1, 2, \ldots, d\}$ with $|J| = k$ and its partition $J = J_0 \cup J_1$ so that the face is given as

$$Q \cap H_{J_0}(0) \cap H_{J_1}(1). \tag{2}$$

Given a set S, its dual cone is defined as $S^* := \{x : x^{\mathrm{T}} s \geqslant 0,\ \ s \in S\}$. Let L be a linear map. Then, it is easy to observe that

$$y \in (LS)^* \quad \Leftrightarrow \quad L^{\mathrm{T}} y \in S^*. \tag{3}$$

It is well known that when S is polyhedral, S^* is generated by the vectors determining the facets of S. Hence, we have

$$Q^* = \mathrm{cone}\{e_1, \ldots, e_d, f_1, \ldots, f_d\}, \tag{4}$$

where e_i denotes the ith unit vector and $f_i := e_0 - e_i$. Let $K_1 \subseteq Q$ and $K_2 \subseteq Q$ be convex cones such that $K = K_1 \cap K_2$. For instance, if K is polyhedral, then K_1 and K_2 can be obtained by taking proper subsystems of

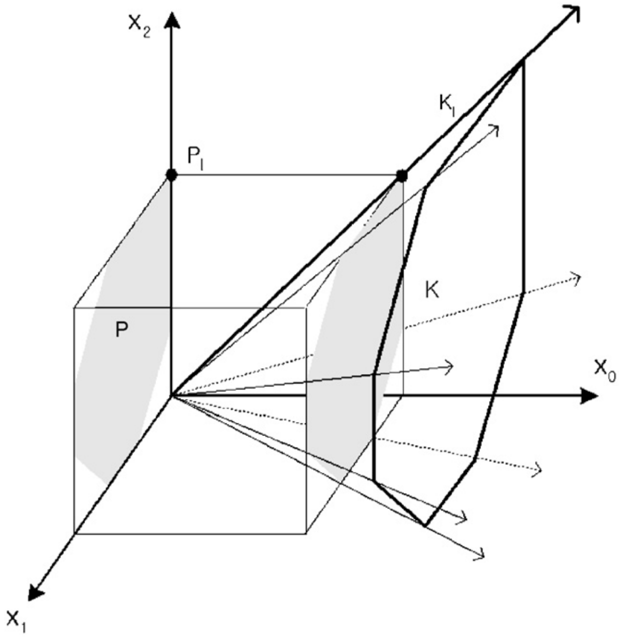

Fig. 1. P, P_{I}, K, and K_{I}.

the linear systems determining K. We are ready to define the lift-and-project operators N_0, N, and N_+ in increasing strength. For $Y \in \mathbb{R}^{(d+1)\times(d+1)}$, consider the conditions:

$$\mathrm{diag}(Y) = Ye_0, \tag{5}$$

$$u^{\mathrm{T}} Y v \geqslant 0 \quad \forall u \in K_1^*, \quad v \in K_2^*, \tag{6}$$

where $\mathrm{diag} : \mathbb{R}^{(d+1)\times(d+1)} \to \mathbb{R}^{d+1}$ maps the diagonal elements of the given matrix onto a vector. Then

$$M_0(K_1, K_2) := \{Y = (y_{ij})_{i,j\in\{0,1,\ldots,d\}} : Y \text{ satisfies (5) and (6)}\}.$$

Notice that (5) and (6), respectively, can be restated as follows:

$$\langle Y, e_i f_i^{\mathrm{T}} \rangle := \mathrm{trace}(Y^{\mathrm{T}} e_i f_i^{\mathrm{T}}) = 0 \quad \forall i \in \{1, 2, \ldots, d\}, \tag{7}$$

$$Y K_2^* \subseteq (K_1^*)^* = K_1. \tag{8}$$

The additional condition

$$Y \in \Sigma^{d+1}, \quad \text{the } (d+1) \times (d+1) \text{ symmetric matrices}, \tag{9}$$

yields the stronger operator

$$M(K_1, K_2) := \{Y \in M_0(K_1, K_2) : Y \text{ satisfies (9)}\}.$$

An additional positive semidefiniteness constraint

$$Y \in \Sigma_+^{d+1}, \quad \text{the } (d+1) \times (d+1) \text{ PSD matrices}, \tag{10}$$

gives

$$M_+(K_1, K_2) := \{Y \in M(K_1, K_2) : Y \text{ also satisfies (10)}\}.$$

We use $N_\sharp \in \{N_0, N, N_+\}$ and $M_\sharp \in \{M_0, M, M_+\}$ to state definitions and results for all three operators M_0, M, M_+, and N_0, N, N_+ (defined below), respectively:

$$N_\sharp(K_1, K_2) := \{Ye_0 : Y \in M_\sharp(K_1, K_2)\}. \tag{11}$$

$N_\sharp(K_1, K_2)$ is a relaxation of K_{I} tighter than K. We have

$$K \supseteq N_0(K_1, K_2) \supseteq N(K_1, K_2) \supseteq N_+(K_1, K_2) \supseteq K_{\mathrm{I}}. \tag{12}$$

When $K_1 := K$, we can use for K_2 any convex cone such that $K \subseteq K_2 \subseteq Q$. While the choice $K_2 := K$ provides the tightest relaxations, the simplicity of Q (especially of Q^*) allows the usage of more elegant and simpler mathematical tools. Moreover, choosing $K_2 := Q$ yields a sequence of clearly tractable relaxations from a computational complexity point of view as we explain below. In this case, by (4), (8) is equivalent to

$$Ye_i, Yf_i \in K, \quad i \in \{1, 2, \ldots, d\}. \tag{13}$$

For this case, we will adopt the following notation:

$$M_\sharp(K) := M_\sharp(K, Q), \quad N_\sharp(K) := N_\sharp(K, Q). \tag{14}$$

Clearly, $N_\sharp$ operators can be applied iteratively:

$$K := N_\sharp^0(K), \quad N_\sharp^t(K) := N_\sharp(N_\sharp^{t-1}(K)) \text{ for } t \in \{1, 2, \ldots\}. \tag{15}$$

The following conventional notation is also useful:

$$N_\sharp(P) := \left\{x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} \in N_\sharp(K)\right\}. \tag{16}$$

Now, we review various facts on the lift-and-project methods:

$$N_0(K) = \bigcap_{i=1}^{d} [(K \cap H_i(0)) + (K \cap H_i(1))]. \tag{17}$$

For a given set J with $|J| = t$, consider the union $\bar{F}_{d+1-t}(J)$ of the $(d+1-t)$-dimensional faces of Q determined by the partitions of J in (2):

$$\bar{F}_{d+1-t}(J) := \bigcup_{J_0 \cup J_1 = J} Q \cap H_{J_0}(0) \cap H_{J_1}(1).$$

Then we can define the following operator:

$$\tilde{N}_0^t(K) := \bigcap_{J \subseteq N, |J| = t} \text{cone } (K \cap \bar{F}_{d+1-t}(J)). \tag{18}$$

Then, the $N_\sharp$-operators have the following relations:

$$N_+^t(K) \subseteq N^t(K) \subseteq N_0^t(K) \subseteq \tilde{N}_0^t(K). \tag{19}$$

Since $\tilde{N}_0^d(K) = K_{\mathrm{I}}$, the above fact implies that the lift-and-project procedures capture the integral hull in at most d iterations. A remarkable fact is that linear optimization on $N_\sharp(K)$ can be done in polynomial time if K is polynomially separable. It can be shown that (5), (9), (10), and (13) are polynomially separable constraints if K is so. Roughly speaking, a separation of $N_\sharp^t(K)$ requires the separation of $N_\sharp^{t-1}(K)$, $\mathrm{O}(d)$ times. This implies that $N_\sharp^t(K)$ is polynomially separable when $t = \mathrm{O}(1)$.

We conclude this section with an elementary fact. For any $0 < \lambda < 1$, and sets $A, B \subseteq \mathbb{R}^d$, consider the following binary operation: $\lambda A + (1-\lambda)B := \{\lambda x + (1-\lambda)y : x \in A, y \in B\}$.

Proposition 1 (*"Concavity"*). *$M_\sharp$ and $N_\sharp$ are concave set operators: For every convex set $P_1, P_2 \subseteq [0,1]^d$ and their homogenizations K_1, K_2, we have*

$$\lambda M_\sharp(K_1) + (1-\lambda)M_\sharp(K_2) \subseteq M_\sharp(\lambda K_1 + (1-\lambda)K_2) \quad \textit{and} \tag{20}$$

$$\lambda N_\sharp(K_1) + (1-\lambda)N_\sharp(K_2) \subseteq N_\sharp(\lambda K_1 + (1-\lambda)K_2). \tag{21}$$

Proof. It suffices to prove (20) as it implies (21). Suppose $Y \in \lambda M_\sharp(K_1) + (1-\lambda)M_\sharp(K_2)$. Then, $Y = \lambda Y_1 + (1-\lambda)Y_2$ for some Y_i, $i \in \{1,2\}$ that satisfy (5), (9), (10), and (13) for K_i, $i \in \{1,2\}$, respectively. Then, it is easy to observe that Y also satisfies the same conditions for $\lambda K_1 + (1-\lambda)K_2$. Therefore, $Y \in M_\sharp(\lambda K_1 + (1-\lambda)K_2)$. □

3. Lower-bound analysis

We mentioned at the end of the last section that after O(1) iterations of $N_\sharp$ operator, the resulting relaxation $N_\sharp^t(K)$ of K_I is still tractable, provided K is polynomially separable. In both theory and practice of combinatorial optimization, it is extremely important to come up with tight relaxations (good outer approximations) of K_I that are tractable. Therefore, a most natural and important question regarding the lift-and-project procedures is what is the smallest number of iterations, t, required to find the integral hull of a combinatorial optimization problem. Establishing this smallest number t is usually done in two parts:

- proving $N_\sharp^t(K) \subseteq K_\mathrm{I}$,
- proving $\exists v \in N_\sharp^{(t-1)}(K) \backslash K_\mathrm{I}$.

These two parts are based on very different mathematical techniques and it is part 2 that seems much harder and much less unified. In this paper, we focus on this second part, establishing lower bounds on the smallest number t. Lower bounds have been established for several problems [10,16,30] and we describe a framework unifying these analyses.

3.1. $N_\sharp$-ranks

Let Π be a 0-1 integer programming problem with instances $\imath$. Denote the input size of $\imath$ by $\langle \imath \rangle$ and $\Pi_n := \{\imath \in \Pi : \langle \imath \rangle \leqslant n\}$. The rank r is a function on the quadruples $(N_\sharp, \Pi, P, n)$, where P is an initial relaxation scheme of instances $\imath$ of Π. For each $\imath$, let $P(\imath) \subseteq Q$ be the relaxation obtained by P applied to $\imath$, and $\ell_\imath$ the minimum ℓ such that $N_\sharp^\ell(P(\imath)) \subseteq P_\mathrm{I}(\imath)$, the integral hull of $P(\imath)$. Then, the rank function r is defined as

$$r(N_\sharp, \Pi, P, n) := \max\{\ell_\imath : \imath \in \Pi_n\}. \tag{22}$$

When Π and n are clear from the context, we will simply write $r_\sharp(P) := r(N_\sharp, \Pi, P, n)$. Obviously, $r_\sharp(P)$ is a measure of efficiency of the lift-and-project methods for problem Π. However, finding an exact value of r is usually a difficult task. Therefore, the analyses are focused on finding good lower and/or upper bounds on $r_\sharp(P)$. The former is equivalent to finding an instance $\imath \in \Pi_n$, a suitable point $v(n)$ and the largest ℓ_n and such that $v(n)$ lies in the gap between $P_\mathrm{I}(\imath)$ and $N_\sharp^{\ell_n}(P(\imath))$: $v(n) \in N_\sharp^{\ell_n}(P(\imath)) \backslash P_\mathrm{I}(\imath)$. Then, clearly $r_\sharp(P) \geqslant \ell_n + 1$. For such an analysis, see also [2,17,22,21].

3.2. Construction of $v(n)$

Many of the existing proofs set up symmetric structures (graphs or polytopes) which allow arguments with convex combinations in the relaxations $N_\sharp^k(P)$. This in turn reduces the number of parameters in $v(n)$.

We denote by $\bar{e}$ the vector of all ones of appropriate size. Suppose $v \in \mathbb{R}_+^d$ maximizes $\bar{e}^\mathrm{T} x$ (we assume for this discussion that the underlying combinatorial optimization problem is a maximum cardinality problem) over $N_\sharp^k(P)$. Thus, if $N_\sharp^k(P)$ is invariant under all permutations $\mathscr{S}_d$ (represented as permutation matrices), i.e.,

$$\forall R \in \mathscr{S}_d : \ x \in N_\sharp^k(P) \quad \Longleftrightarrow \quad Rx \in N_\sharp^k(P),$$

$$\begin{tikzcd}
K_j \arrow[r, "L"] \arrow[d, "M_\sharp"] \arrow[dr, phantom, "\circlearrowright"] & LK_j \arrow[d, "M_\sharp"'] \\
M_\sharp(K_1,K_2) \arrow[r, hook, "L \cdot L^T"] & M_\sharp(LK_1,LK_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
K_j \arrow[r, "L"] \arrow[d, "N_\sharp"] \arrow[dr, phantom, "\circlearrowright"] & LK_j \arrow[d, "N_\sharp"'] \\
N_\sharp(K_1,K_2) \arrow[r, hook, "L"] & N_\sharp(LK_1,LK_2)
\end{tikzcd}$$

$M_\sharp$-commutativity $\qquad$ $N_\sharp$-commutativity

Fig. 2. $M_\sharp$- and $N_\sharp$-commutative diagram.

then

$$\left(\frac{1}{|\mathscr{S}_d|}\sum_{R\in\mathscr{S}_d} Rv\right) \in N_\sharp^k(P),$$

by the convexity of $N_\sharp^k(P)$. Therefore, we can assume $v = \alpha\bar{e}$ for some $\alpha \geqslant 0$ (we used $\bar{e}^{\mathrm{T}}Sv = \bar{e}^{\mathrm{T}}v$, $\forall R \in \mathscr{S}_d$).

In other problems, the symmetry might be less pronounced; however, this basic technique can still be useful in reducing the number of parameters in v from a large function of d to a constant. Then the conditions of $N_\sharp$ may lead to recursions (as in [16]). This kind of technique was used in [30,15,10,25,21,22]. A recent formal approach is presented in [14].

3.3. $M_\sharp$- and $N_\sharp$-commutative maps

An ingredient of our unifying framework is the inductive construction of $v(n)$ of the desired property mentioned in Section 3.1. In doing so, $M_\sharp$- and $M_\sharp$-*commutative maps* are very useful. These maps provide the passage from the space of one induction step (the lower one) to the next.

Definition 2. Suppose $L : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1+k}$ is a linear map. Then, L is said to be $M_\sharp$- and $N_\sharp$-commutative, respectively, if $LM_\sharp(K_1, K_2)L^{\mathrm{T}} \subseteq M_\sharp(LK_1, LK_2)$ and $LN_\sharp(K_1, K_2) \subseteq N_\sharp(LK_1, LK_2)$ for every pair of closed convex cones $K_1, K_2 \subseteq Q$ (see Fig. 2).

Let $\tilde{e}_i$'s and $\tilde{f}_i$'s be the extreme rays of the dual cone $\tilde{Q}^*$ of the $(d + 1 + k)$-dimensional cone $\tilde{Q}$ spanned by the $(d + k)$-dimensional hypercube.

Theorem 3. *A linear map $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$ is $M_\sharp$-commutative if and only if, for every $j \in \{1, 2, \ldots, d+1+k\}$,*

$$L^{\mathrm{T}}\tilde{e}_j\tilde{f}_j^{\mathrm{T}}L \in \operatorname{span}\{e_i f_i^{\mathrm{T}} : i \in \{1, 2, \ldots, d\}\}. \tag{23}$$

Proof. Assume $Y \in M_\sharp(K_1, K_2)$. We first show that (23) guarantees $LYL^{\mathrm{T}} \in M_\sharp(LK_1, LK_2)$. First, notice that (6), (9), and (10) are true for LYL^{T} regardless of (23): (9) and (10) are clearly satisfied by LYL^{T}. Regarding (6), due to (3), $\tilde{w} \in (LK_j)^*$ if and only if $L^{\mathrm{T}}\tilde{w} \in K_j^*$ for $j \in \{1, 2\}$. Therefore, $Y \in M_\sharp(K_1, K_2)$ implies $0 \leqslant (L^{\mathrm{T}}\tilde{u})^{\mathrm{T}}Y(L^{\mathrm{T}}\tilde{v}) = \tilde{u}^{\mathrm{T}}LYL^{\mathrm{T}}\tilde{v}$ for any $\tilde{u} \in (LK_1)^*$and $\tilde{v} \in (LK_2)^*$. It remains to show (23) guarantees (5) for LYL^{T}.

However, by (7) the latter is equivalent to that for all $j \in \{1, 2, \ldots, d + k\}$,

$$\operatorname{trace}(LYL^{\mathrm{T}}\tilde{e}_j\tilde{f}_j^{\mathrm{T}}) = \operatorname{trace}(Y(L^{\mathrm{T}}\tilde{e}_j\tilde{f}_j^{\mathrm{T}}L)) = 0. \tag{24}$$

From (23), $L^{\mathrm{T}}\tilde{e}_j\tilde{f}_j^{\mathrm{T}}L = \sum_i \lambda_i e_i f_i^{\mathrm{T}}$ for some λ_i, $i \in \{1, 2, \ldots, d\}$. Since Y satisfies (7), this implies (24).

We will prove the necessity for the M-operator. The proofs for M_0- and M_+-operators are similar. Suppose (23) does not hold for $j = 1$: $L^{\mathrm{T}}\tilde{e}_1\tilde{f}_1^{\mathrm{T}}L \notin \operatorname{span}\{e_i f_i^{\mathrm{T}} : 1, 2, \ldots, d\}$. Then, there is $\bar{Y} \in \Sigma^{d+1}$ such that $\operatorname{trace}(\bar{Y}e_i f_i^{\mathrm{T}}) = 0$ for all $i \in \{1, 2, \ldots, d\}$ and $\operatorname{trace}(\bar{Y}L^{\mathrm{T}}\tilde{e}_1\tilde{f}_1^{\mathrm{T}}L) \neq 0$. Pick K_1 and K_2 so that $M(K_1, K_2)$ is full-dimensional in

$\{Y \in \Sigma^{d+1} : \mathrm{diag}(Y) = Ye_0\}$. Let $\hat{Y} \in \mathrm{rel\ int}(M(K_1, K_2))$. Then, there is $\bar{\varepsilon} > 0$such that $\hat{Y} + \varepsilon\bar{Y} \in M(K_1, K_2)$ for all $\varepsilon < \bar{\varepsilon}$.

But, since $\mathrm{trace}(\bar{Y} L^{\mathrm{T}} \tilde{e}_1 \tilde{f}_1^{\mathrm{T}} L) \neq 0$, $\mathrm{trace}(L(\hat{Y} + \varepsilon\bar{Y})L^{\mathrm{T}} \tilde{e}_1 \tilde{f}_1^{\mathrm{T}}) = \mathrm{trace}(\hat{Y} L^{\mathrm{T}} \tilde{e}_1 \tilde{f}_1^{\mathrm{T}} L) + \varepsilon \mathrm{trace}(\bar{Y} L^{\mathrm{T}} \tilde{e}_1 \tilde{f}_1^{\mathrm{T}} L)$ cannot be identically 0 on $0 < \varepsilon < \bar{\varepsilon}$. Hence, L is not M-commutative. □

Remark 4. Notice that in Theorem 3, k is not necessarily assumed to be nonnegative.

Corollary 5. *If, in addition, L is invertible, then the equality holds:* $LM_\sharp(K_1, K_2)L^{\mathrm{T}} = M_\sharp(LK_1, LK_2)$.

Proof. It suffices to show that if $\tilde{Y} \in M_\sharp(LK_1, LK_2)$, then $L^{-1}\tilde{Y}L^{-\mathrm{T}} \in M_\sharp(K_1, K_1)$, namely $L^{-1}\tilde{Y}L^{-\mathrm{T}}$ satisfies (6), (7), (9), and (10). Clearly, (9) and (10) are satisfied. Also, from (3) and invertibility of L, it follows that $(LK_i)^* = L^{-\mathrm{T}}K_i^*$ for $i \in \{1, 2\}$. Therefore, $\tilde{u}^{\mathrm{T}}\tilde{Y}\tilde{v} \geqslant 0$ for all $\tilde{u} \in L^{-\mathrm{T}}K_1^*$ and $\tilde{v} \in L^{-\mathrm{T}}K_2^*$, or equivalently $u^{\mathrm{T}}L^{-1}\tilde{Y}L^{-\mathrm{T}}v \geqslant 0$ for all $u \in K_1^*$ and $v \in K_2^*$.

Finally, to prove (7), since $\mathrm{trace}(L^{-1}\tilde{Y}L^{-\mathrm{T}}e_i f_i^{\mathrm{T}}) = \mathrm{trace}(\tilde{Y}L^{-\mathrm{T}}e_i f_i^{\mathrm{T}}L^{-1})$, it suffices to show that the linear subspace $\mathrm{span}\{e_i f_i^{\mathrm{T}}\}$ is invariant under the mapping $L^{-\mathrm{T}} \cdot L^{-1}$. Or equivalently, $L^{\mathrm{T}}\,\mathrm{span}\{e_i f_i^{\mathrm{T}}\}L = \mathrm{span}\{e_i f_i^{\mathrm{T}}\}$. But, since L is invertible, the linear map $L^{\mathrm{T}} \cdot L : \mathrm{span}\{e_i f_i^{\mathrm{T}}\} \to \mathrm{span}\{e_i f_i^{\mathrm{T}}\}$ is a bijection. Hence, the corollary follows. □

Corollary 6. *If L and L' are $M_\sharp$-commutative maps, then their composite, if defined, is also $M_\sharp$-commutative.*

The following facts were observed by Lovász and Schrijver.

Lemma 7 (*Lovász and Schrijuer [26]*). *If L is $M_\sharp$-commutative and $L^{\mathrm{T}}e_0$ is parallel to e_0, then L is also $N_\sharp$-commutative.*

Corollary 8 (*Lovász and Schrijuer [26]*). *If $L : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is an automorphism of Q, namely a linear map such that $LQ = Q$, then for every pair of closed convex cones $K_1, K_2 \subseteq Q$, we have $LM_\sharp(K_1, K_2)L^{\mathrm{T}} = M_\sharp(LK_1, LK_2)$ and $LN_\sharp(K_1, K_2) = N_\sharp(LK_1, LK_2)$.*

The proof follows from Corollary 5 and Lemma 7. For if L is an automorphism of Q then there are a permutation $\sigma : \{1, 2, \ldots, d\} \to \{1, 2, \ldots, d\}$ and $\lambda > 0$ such that $\{L^{\mathrm{T}}e_i, L^{\mathrm{T}}f_i\} = \{\lambda e_{\sigma(i)}, \lambda f_{\sigma(i)}\}$, and $L^{\mathrm{T}}e_0$ is parallel to e_0. For the details, the reader is referred to the Appendix.

Notice that in Definition 2 $M_\sharp$- and $N_\sharp$-commutative linear maps that are not necessarily assumed to be invertible. The following are such examples of $M_\sharp$- and $N_\sharp$-commutative maps:

- *Embedding* $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$ so that, for some $0 \leqslant l \leqslant k$,

$$\tilde{x}_i := \begin{cases} x_i & \text{for } i \in \{0, 1, \ldots, d\}, \\ 0 & \text{for } i \in \{d+1, \ldots, d+l\}, \\ x_0 & \text{for } i \in \{d+l+1, \ldots, d+k\}. \end{cases} \tag{25}$$

- *Duplication* $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$ so that, for a subset $\{j_1, \ldots, j_k\} \subseteq \{1, 2, \ldots, d\}$,

$$\tilde{x}_i := \begin{cases} x_i & \text{for } i \in \{0, 1, \ldots, d\}, \\ x_{j_{i-d}} & \text{for } i \in \{d+1, \ldots, d+k\}. \end{cases} \tag{26}$$

- *Flipping* is an automorphism that maps $e_j \mapsto f_j$, $f_j \mapsto e_j$ for each $j \in J \subseteq \{1, 2, \ldots, d\}$.

Indeed, in all of the above examples, one can check that for every $j \in \{1, 2, \ldots, d+k\}$, there is $i \in \{1, 2, \ldots, d\}$ such that

$$\{L^{\mathrm{T}}\tilde{e}_j, L^{\mathrm{T}}\tilde{f}_j\} = \{e_0, 0\} \text{ or } \{e_i, f_i\}, \tag{27}$$

which is sufficient for (23). In fact, (27) describes a fairly broad class of linear maps that are both $M_\sharp$- and $N_\sharp$-commutative.

Corollary 9. *Suppose L satisfies the following conditions*: (1) *The first row is* e_0 *and* (2) *the rest are either*, 0, e_0, e_i, *or* f_i *for* $i \in \{1, 2, \dots, d\}$. *Then any positive multiple of L is both* $M_\sharp$- *and* $N_\sharp$-*commutative.*

Now, we discuss one of the key properties used in our framework for lower-bound analysis.

Lemma 10. *Let* $K \subseteq \mathbb{R}^{d+1}$ *and* $\tilde{K} \subseteq \mathbb{R}^{d+1+k}$, *respectively, be the homogenizations of the convex sets* $P \subseteq [0,1]^d$ *and* $\tilde{P} \subseteq [0,1]^{d+k}$. *Assume* $L : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1+k}$ *is an* $N_\sharp$-*commutative map. If L is feasible, namely* $LK \subseteq \tilde{K}$, *then for every* $t \geqslant 0$, *we have* $LN_\sharp^t(K) \subseteq N_\sharp^t(LK) \subseteq N_\sharp^t(\tilde{K})$.

Proof. By induction on t using the feasibility of L. □

The following fact is potentially useful in a lower-bound analysis.

Theorem 11. *Every convex combination of* $N_\sharp$-*commutative maps is also* $N_\sharp$-*commutative.*

Proof. Let L_1 and L_2 be $N_\sharp$-commutative maps and $0 < \lambda < 1$. Define $L := \lambda L_1 + (1-\lambda)L_2$. Then,

$$\begin{aligned} LN_\sharp(K) &= \lambda L_1 N_\sharp(K) + (1-\lambda)L_2 N_\sharp(K) \\ &\subseteq \lambda N_\sharp(L_1 K) + (1-\lambda)N_\sharp(L_2 K) \\ &\subseteq N_\sharp(\lambda L_1 K + (1-\lambda)L_2 K) = N_\sharp(LK). \end{aligned}$$

$N_\sharp$-commutativity of L_1 and L_2 implies the first inclusion. The second inclusion follows from the concavity of $N_\sharp$-operators. □

We note that $M_\sharp$-commutativity, however, is not necessarily preserved under taking convex combinations.

Analogously to the relation of P and K, for the notation of (16), we can also define

$$LP := \left\{ x : \begin{pmatrix} 1 \\ x \end{pmatrix} \in LK \right\}, \quad LN_\sharp(P) := \left\{ x : \begin{pmatrix} 1 \\ x \end{pmatrix} \in LN_\sharp(K) \right\}. \tag{28}$$

If L is $N_\sharp$-commutative, then it is routine to check that

$$LN_\sharp(P) \subseteq N_\sharp(LP). \tag{29}$$

3.4. Unifying approach

Most lower-bound analyses rely on a mathematical induction on the size (suitably defined) of the instances. To facilitate the presentation, we consider only the instances that are symmetric with respect to the variables. Thus, we consider essentially a unique instance of each size. Let s_k be the size of the instance at the kth induction step. For instance, s_k can be the number of edges, nodes, or variables. Denote by P_{s_k} and K_{s_k}, respectively, the initial relaxation and its homogenization for Π_{s_k}. For simplicity, we will write $M_\sharp(s_k) := M_\sharp(K_{s_k})$ and $N_\sharp(s_k) := N_\sharp(K_{s_k})$. The unifying approach focuses on constructing in a recursive manner, the sequence of proofs $Y_k \in M_\sharp^k(s_k)$ such that $v(k) = Y_k e_0$ via appropriate $M_\sharp$-commutative maps L_p, see Fig. 3.

Scheme 12. *Using the symmetry of* $\imath$, P, *and* $v(k)$, *construct* $\{Y_k\}$ *so that* $Y_k e_0 = v(k)$, $Y_k \in M_\sharp^k(s_k)$ *and* $Y_{k+1} = (1/|S|)\sum_{p \in S} L_p Y_k L_p^{\mathrm{T}} \in M_\sharp^{k+1}(s_{k+1})$, *for some set of* $M_\sharp$-*commutative maps* $\{L_p : p \in S\}$.

The $M_\sharp$-commutativity of L_p's implies that $L_p Y_k L_p^{\mathrm{T}} \in M_\sharp^k(s_{k+1})$ for all p. Thus, the scheme is based on the intuition that, due to the symmetry, when $L_p Y_k L_p^{\mathrm{T}} \in M_\sharp^k(s_{k+1})$ for $p \in S$ then their convex combination might lie in the smaller set $M_\sharp^{k+1}(s_{k+1})$. See Fig. 4.

$$
\begin{tikzcd}
\bullet v(k) \in N^k_\sharp(s_k) \arrow[d, Leftrightarrow] & \bullet w^p \in N^k_\sharp(s_{k+1}) \arrow[d, Leftrightarrow] \\
Y_k \in M^k_\sharp(s_k) \arrow[r, "L_p \cdot L_p^T"] \arrow[d, "\text{induction}"] & W^p = L_p Y_k L_p^T \in M^k_\sharp(s_{k+1}) \arrow[dl, "\text{proof}"] \\
\bullet v(k+1) \in N^{k+1}_\sharp(s_{k+1}) \arrow[d, Leftrightarrow] & \\
\exists ?\ Y_{k+1} \in \operatorname{cone}\{W^p\}. &
\end{tikzcd}
$$

$$\text{s.t } Y_{k+1}e_i,\ Y_{k+1}f_i \in \operatorname{cone}\{w^p\}.$$

Fig. 3. Unifying approach.

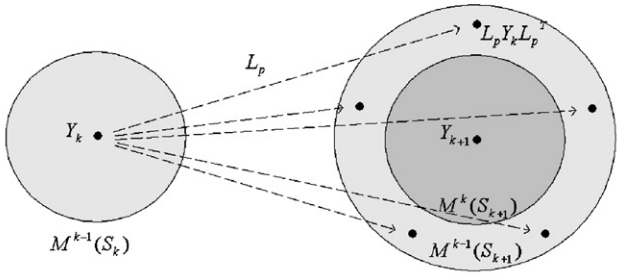

Fig. 4. A schematic illustration of the proof technique.

Note that the convex combination of the commutative maps, $\sum_{p\in S} L_p Y_k L_p^{\mathrm{T}}$, preserves (5), (9), and most importantly, positive semidefiniteness (10) of Y_k, Hence, due to the manner in which Y_k is defined, the conditions are automatically met once we establish them in the base step of the induction. Thus, the unifying approach can make the proof more straightforward and systematic.

This approach can be extended to the special structures that a given problem Π may have. Suppose, for instance, some set of integral points $\{z_q : q \in T\} \subset P_{\mathrm{I}}$ is readily available. Then, for every $q \in T$ and $k \geqslant 0$, $\binom{1}{z_q}\binom{1}{z_q}^{\mathrm{T}} \in M^k_\sharp(K)$. (An interesting special case is when P is upper or lower comprehensive.) Thus, in such cases, the recursive definition of Y_k can be generalized as follows:

$$Y_{k+1} = \mu\left(\frac{1}{|S|}\sum_{p\in S} L_p Y_k L_p^{\mathrm{T}}\right) + (1-\mu)\left(\frac{1}{|T|}\sum_{q\in T}\binom{1}{z_q}\binom{1}{z_q}^{\mathrm{T}}\right), \tag{30}$$

for some appropriate set of integral points $\{z_q : q \in T\}$ and $0 \leqslant \mu \leqslant 1$.

4. Implementations of the unifying proof

4.1. Matching polytope

The matching polytope of a graph $G = (V, E)$ is defined to be the convex hull of the characteristic vectors of the matchings in G. Then, it is the integral hull, P_{I} of

$$P := \{x \in \mathbb{R}^E : x(\delta(v)) \leqslant 1\ \forall v \in V, x \geqslant 0\}, \tag{31}$$

where $\delta(v)$ is the set of edges that have v as an endpoint and $x(S) := \sum_{j\in S} x_j$. Denote by G_{2k+1} the clique with $s_k := (2k+1)$ nodes, $V = \{1, 2, \ldots 2k+1\}$. Then, it has $t_k := k(2k+1)$ edges, $(1, 2), (1, 3), \ldots, (2k, 2k+1)$. Consider the lexicographic order $\prec$ on the edges: $(i, j) \prec (k, l) \Leftrightarrow i < k$, or $i = k$ and $j < l$. We assume that the edges

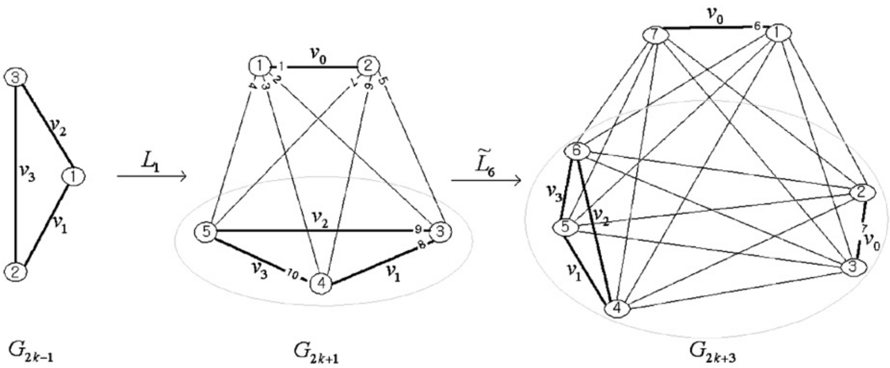

Fig. 5. Two-level embed $\tilde{L}L$.

are numbered to the lexicographic order and denote for simplicity $E_{2k+1} = \{1, 2, \ldots, t_k\}$. See, e.g., G_{2k+1} in Fig. 5 and the numbers assigned to the edges. Also, let P_{2k+1} be the relaxation of (31) for G_{2k+1} and K_{2k+1} the homogenization of P_{2k+1}. Recall that we write $N_+^t(2k+1) := N_+^t(K_{2k+1})$.

Stephen and Tunçel [30] showed that if K_{2k+1} is used as the initial cone, $r_+(P_{2k+1})$ is k. In doing so, they established the lower-bound $(k-1) < r_+(P_{2k+1})$ by constructing a uniform point $v(k)$ in $N_+^{k-1}(2k+1)\backslash(P_{2k+1})_{\mathrm{I}}$: For $k \in \{1, 2, \ldots\}$,

$$v(k) := \begin{pmatrix} 1 \\ \frac{1}{2k}\bar{e} \end{pmatrix} \in \mathbb{R}^{E_{2k+1}\cup\{0\}}. \tag{32}$$

Since the maximum cardinality of a matching on G_{2k+1} is k, $v(k)$ is not in $(P_{2k+1})_{\mathrm{I}}$. Hence, the lower-bound k on $r_+(P_{2k+1})$ will follow, if we show, for $k \in \{1, 2, \ldots\}$,

$$v(k) \in N_+^{k-1}(2k+1). \tag{33}$$

Denote by $G_{2k+1}\backslash p$ the graph obtained by deleting the two endpoints of p from G_{2k+1}. Then, a key observation is that for any $p \in E_{2k+1}$, the lexicographic orders on E_{k-1} and $E(G_{2k+1}\backslash p)$ induce an obvious isomorphism between G_{2k-1} and $G_{2k+1}\backslash p$. Hence, the following map aligns the order of elements of a vector v_i, $i \in E_{2k-1}$, to the order of edges of the sub-clique $G_{2k+1}\backslash p$. For $p \in \{1, 2, \ldots, t_k\}$, define $L_p : v \in \mathbb{R}^{E_{2k-1}\cup\{0\}} \mapsto w^p \in \mathbb{R}^{E_{2k+1}\cup\{0\}}$:

$$w_j^p := \begin{cases} v_0, & j = 0, \\ v_0, & j = p, \\ 0, & j \in \mathrm{Inc}(p), \text{ the set of edges incident to } p, \\ v_i, & j \text{ is the } i\text{th edge of } G_{2k+1}\backslash p. \end{cases} \tag{34}$$

For $p \in E_{2k+1}$ and $q \in E_{2k+3}$, we can define composite $\tilde{L}_q L_p$ of the two embeddings, $L_p : \mathbb{R}^{E_{2k-1}\cup\{0\}} \to \mathbb{R}^{E_{2k+1}\cup\{0\}}$ and $\tilde{L}_q : \mathbb{R}^{E_{2k+1}\cup\{0\}} \to \mathbb{R}^{E_{2k+3}\cup\{0\}}$. In Fig. 5, $\tilde{L}_6 L_1 v$ with $v = (v_0, v_1, v_2, v_3)^{\mathrm{T}}$ is illustrated. Notice that for a given vector $v \in \mathbb{R}^{E_{2k-1}\cup\{0\}}$, there can be more than one two-level embeddings mapping v to the same vector.

Lemma 13. *Let p and $q \in E_{2k+3}$ be two nonincident edges of G_{2k+3}. Suppose p is the p_qth edge of $G_{2k+3}\backslash q$. Similarly, define q_p for q with respect to p. Then for every $v \in \mathbb{R}^{E_{2k-1}\cup\{0\}}$,*

$$\tilde{L}_q L_{p_q} v = \tilde{L}_p L_{q_p} v.$$

Proof. Trivially, by definition, both $\tilde{L}_q L_{p_q} v$ and $\tilde{L}_p L_{q_p} v$ have the same 0th element, v_0.

Since p and q are not incident in G_{2k+3}, $G_{2k+3}\backslash p$, and $G_{2k+3}\backslash q$ have a common sub-clique with $t_{k-1} := (k-1)(2k-1)$ edges, namely $G_{2k+3}\backslash\{p, q\}$. We will show that $\tilde{L}_q L_{p_q}$ assigns (1) v_0 to p and q, (2) for each $i \in E_{2k-1}$, v_i to the ith edge of $G_{2k+3}\backslash\{p, q\}$, and (3) 0's to the remaining edges in E_{2k+3}. Then, by the symmetry between p and q, $\tilde{L}_p L_{q_p}$ does the same and hence the lemma will follow.

By the definition, (34), and the discussion preceding it, L_{p_q} aligns $v_i, i \in E_{2k-1}$ to $E(G_{2k+1}\backslash p_q)$. Now, $\tilde{L}_q$ assigns v_0 to $q \in E_{2k+3}$ and aligns $(L_{p_q}v)_j, j \in E_{2k+1}$ to the edges, $E(G_{2k+3}\backslash q)$. The p_qth element of $L_{p_q}v$ is v_0 and, from the hypothesis, the p_qth element of $E(G_{2k+3}\backslash q)$ is p. Therefore, $\tilde{L}_q L_{p_q}$ assigns v_0 to p. Hence (1) holds. Furthermore, in this alignment $(L_{p_q}v)_j, j \in E_{2k+1}\backslash p_q$ will be assigned to the edges of $(G_{2k+3}\backslash q)\backslash p = G_{2k+3}\backslash\{p, q\}$ preserving the order. But, $(L_{p_q}v)_j, j \in E_{2k+1}\backslash p_q$ are ordered the same as $v_i, i \in E_{2k-1}$. Therefore, (2) also holds. Clearly, the remaining edges of E_{2k+3} are assigned 0's. Hence the lemma. □

For example, in Fig. 5, let $p = 7$ and $q = 6$. Then, $p_q = 1$ and $q_p = 4$. Therefore, another two-level embedding $\tilde{L}_7 L_4 v$, as easily checked, has the same effect as $\tilde{L}_6 L_1 v$.

Now, we provide a proof of (33) based on the unifying approach.

Proof. Clearly, $v(1) \in N_+^0(3) := K_3$ as it satisfies the inequalities of (31). Define $Y_k \in \mathbb{R}^{E_{2k+1}\cup\{0\}\times E_{2k+1}\cup\{0\}}$ recursively as follows:

$$Y_2 := \left[\frac{1}{s_2}\sum_{p=1}^{s_2} L_p v(1);\quad \frac{1}{4}L_1 v(1);\quad \ldots;\quad \frac{1}{4}L_{s_2}v(1)\right], \tag{35}$$

$$Y_k := \frac{1}{t_k}\sum_{p=1}^{t_k} L_p Y_{k-1} L_p^{\mathrm{T}}, \quad k \in \{3, 4, \ldots\}. \tag{36}$$

First, let us prove that, for every $k \in \{2, 3, \ldots\}$,

$$Y_k e_0 = v(k) \quad \text{and} \tag{37}$$

$$Y_k \text{ satisfies (5), (9), and (10).} \tag{38}$$

It is easily seen from (32) and (35) that Y_2 is given as follows:

$$Y_2 = \begin{bmatrix}
1 & \frac14 & \frac14 & \frac14 & \frac14 & \frac14 & \frac14 & \frac14 & \frac14 & \frac14 & \frac14 \\
\frac14 & \frac14 & 0 & 0 & 0 & 0 & 0 & 0 & \frac14 & \frac14 & \frac14 \\
\frac14 & 0 & \frac14 & 0 & 0 & 0 & \frac14 & \frac14 & 0 & 0 & \frac14 \\
\frac14 & 0 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 \\
\frac14 & 0 & 0 & 0 & \frac14 & \frac14 & \frac14 & 0 & \frac14 & 0 & 0 \\
\frac14 & 0 & 0 & \frac14 & \frac14 & \frac14 & 0 & 0 & 0 & 0 & \frac14 \\
\frac14 & 0 & \frac14 & 0 & \frac14 & 0 & \frac14 & 0 & 0 & \frac14 & 0 \\
\frac14 & 0 & \frac14 & \frac14 & 0 & 0 & 0 & \frac14 & \frac14 & 0 & 0 \\
\frac14 & \frac14 & 0 & 0 & \frac14 & 0 & 0 & \frac14 & \frac14 & 0 & 0 \\
\frac14 & \frac14 & 0 & \frac14 & 0 & 0 & \frac14 & 0 & 0 & \frac14 & 0 \\
\frac14 & \frac14 & \frac14 & 0 & 0 & \frac14 & 0 & 0 & 0 & 0 & \frac14
\end{bmatrix}. \tag{39}$$

Thus, Y_2 satisfies (37). Now, we show it is also true for $k + 1$. But,

$$\begin{aligned}
Y_{k+1}\tilde{e}_0 &= \frac{1}{t_{k+1}}\sum_{q=1}^{t_{k+1}} \tilde{L}_q Y_k \tilde{L}_q^{\mathrm{T}} \tilde{e}_0 = \frac{1}{t_{k+1}}\sum_{q=1}^{t_{k+1}} \tilde{L}_q Y_k e_0 \\
&= \frac{1}{t_{k+1}}\sum_{q=1}^{t_{k+1}} \tilde{L}_q v(k) = \frac{1}{t_{k+1}}\begin{pmatrix} t_{k+1} \\ (1 + t_k \frac{1}{2k})e \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2(k+1)}e \end{pmatrix}.
\end{aligned}$$

Thus, (37) holds for $(k+1)$ and hence for all $k \geqslant 2$. Regarding (38), Y_2 clearly satisfies (5) and (9). It is straightforward to check that Y_2 is positive semidefinite: It has eigenvalues 0, $\frac{5}{24}$ and $\frac{19}{12}$. Hence, by induction, M_+-commutativity of the embeddings L_p implies (38).

To complete the proof, due to (8), it suffices to show that for $k \in \{2, 3, \ldots\}$,

$$Y_k e_i,\ Y_k f_i \in N_+^{k-2}(2k+1) \quad \forall i \in \{1, 2, \ldots, t_k\}. \tag{40}$$

To do so, we will prove the following:

$$Y_k e_i = \frac{1}{2k} L_i v(k-1) \quad \text{and} \tag{41}$$

$$Y_k f_i = \frac{1}{4k} \sum_{j \in \mathrm{Inc}(i)} L_j v(k-1). \tag{42}$$

Then, since $v(1) \in K_3 := N_+^0(3)$, using the N_+-commutativity and the feasibility of L_i, it is easy to see inductively that (40) follows from (41) and (42). By definition of Y_2, (41) is satisfied when $k = 2$. Now, we show (41) holds for $k+1$. By definition,

$$Y_{k+1}\tilde{e}_j = \frac{1}{t_{k+1}} \sum_{q=1}^{t_{k+1}} \tilde{L}_q Y_k \tilde{L}_q^{\mathrm{T}} \tilde{e}_j.$$

Case 1: If $q = j$, then $\tilde{L}_q Y_k \tilde{L}_q^{\mathrm{T}} \tilde{e}_j = \tilde{L}_j Y_k e_0 = \tilde{L}_j v(k)$ from (37).

Case 2: If q and j are incident, then $\tilde{L}_q^{\mathrm{T}} \tilde{e}_j = 0$. Therefore, we have $\tilde{L}_q Y_k \tilde{L}_q^{\mathrm{T}} \tilde{e}_j = 0$.

Case 3: Finally, consider the case when q and j are not incident. Suppose j is the j_qth smallest numbered edge of $G_{2k+3}\backslash q$. Then, by the definition of $\tilde{L}$, we have $\tilde{L}_q^{\mathrm{T}} \tilde{e}_j = e_{j_q}$. Therefore, $\tilde{L}_q Y_k \tilde{L}_q^{\mathrm{T}} \tilde{e}_j = \tilde{L}_q Y_k e_{j_q}$, which is $(1/2k)\tilde{L}_q L_{j_q} v(k-1)$ from the induction hypothesis, (41). Suppose q is the q_jth edge of $G_{2k+3}\backslash j$. Then, by Lemma 13, we obtain $(1/2k)\tilde{L}_q L_{j_q} v(k-1) = (1/2k)\tilde{L}_j L_{q_j} v(k-1)$. Clearly, for a fixed j, distinct q's have distinct q_j's from $\{1, 2, \ldots, t_k\}$.

Thus, summarizing the cases, we obtain

$$Y_{k+1}\tilde{e}_j = \frac{1}{t_{k+1}} \left(\tilde{L}_j v(k) + \frac{1}{2k} \sum_{i=1}^{t_k} \tilde{L}_j L_i v(k-1) \right) \tag{43}$$

$$= \frac{1}{t_{k+1}} \tilde{L}_j \left(v(k) + \frac{t_k}{2k} v(k) \right) = \frac{1}{2(k+1)} \tilde{L}_j v(k). \tag{44}$$

The first equality of (44) is from that $\sum_{i=1}^{t_k} L_i v(k-1) = t_k v(k)$. Therefore, (41) holds for $k+1$ and the proof of (41) is completed.

Finally, it is routine to check that (42) is implied by (37) and (41). Thus, (33) follows. □

Remark 14. The use of two-level embedding in the above proof can be avoided (making the proof considerably shorter) by a careful counting argument. However, the above linear algebraic proof via the commuting single-level embeddings may be useful in other more complicated situations. Aguilera et al. [1] gave another proof using the work of Doob [11] which in turn has connections to Tutte's much earlier work [31]. While this is a very nice connection found by [1], the underlying proof still relies on working out the eigenspaces and the eigenvalues of the corresponding Y matrix to establish the positive semidefiniteness.

4.2. *Knapsack polytope*

Consider the following (reversed) knapsack polytope and its 0-1 integral hull:

$$P_d := \{x \in \mathbb{R}^d : x_1 + \cdots + x_d \geqslant \tfrac{1}{2}\}, \quad (P_d)_{\mathrm{I}} = \{x \in \mathbb{R}^d : x_1 + \cdots + x_d \geqslant 1\}. \tag{45}$$

Cook and Dash [10] showed that these are some of the worst-case examples for all $N_\sharp$ operators: $r_0(P_d) = r(P_d) = r_+(P_d) = d$. To capture such results, it suffices to show that

$$v(d) := \begin{pmatrix} 1 \\ \frac{1}{d+1}\bar{e} \end{pmatrix} \in N_+^{d-1}(d). \tag{46}$$

We use the following embeddings. For $p \in \{1, 2, \ldots, d\}$, $L_p : v \in \mathbb{R}^d \mapsto w^p \in \mathbb{R}^{d+1}$ such that

$$w_j^p := \begin{cases} v_0, & j = 0, \\ 0, & j = p, \\ v_i, & j \text{ is the } i\text{th smallest number of } \{1, 2, \ldots, d\} \backslash \{p\}. \end{cases} \tag{47}$$

We recursively construct Y_d, for $d \in \{1, 2, \ldots\}$, as follows:

$$Y_1 := \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \tag{48}$$

$$Y_d := \frac{1}{d+1} \sum_{p=1}^{d} L_p Y_{d-1} L_p^{\mathrm{T}} + \frac{1}{d+1} \mathrm{Arrow}_d \left(\frac{1}{d} \bar{e} \right) \quad \text{for } d \in \{2, 3, \ldots\}, \tag{49}$$

where $\mathrm{Arrow}_d(\cdot) : \mathbb{R}^d \mapsto \Sigma^{d+1}$ is defined as

$$\mathrm{Arrow}_d(u) := \begin{bmatrix} 1 & u_1 & u_2 & \cdots & u_d \\ u_1 & u_1 & 0 & \cdots & 0 \\ u_2 & 0 & u_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_d & 0 & 0 & \cdots & u_d \end{bmatrix}.$$

Note that, $\mathrm{Arrow}_d((1/d)\bar{e})$ fits the recursion of (30), since it corresponds to a convex combination (with uniform coefficients $(1/d)$) of the matrices corresponding to known integer points of P:

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Clearly, Y_1 satisfies (5), (9), and (10). Also, we have

$$Y_d = \begin{bmatrix} 1 & \frac{1}{d+1} & \frac{1}{d+1} & \cdots & \frac{1}{d+1} \\ \frac{1}{d+1} & \frac{1}{d+1} & 0 & \cdots & 0 \\ \frac{1}{d+1} & 0 & \frac{1}{d+1} & \ddots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ \frac{1}{d+1} & 0 & 0 & 0 & \frac{1}{d+1} \end{bmatrix}.$$

In particular, $Y_d e_0 = v(d)$. Moreover, $Y_d e_j = (1/(d+1))(e_0 + e_j)$. Hence, $(d+1)Y_d e_j \in P_{\mathrm{I}}(d)$. Finally,

$$Y_d f_j = \frac{d}{d+1}\begin{pmatrix} 1 \\ \frac{1}{d} \\ \vdots \\ \frac{1}{d} \\ 0 \\ \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{pmatrix} = \frac{d}{d+1} L_j v(d-1) \in N_+^{d-2}(d).$$

Thus, $Y_d e_j, Y_d f_j \in N_+^{d-2}(d)$ for all $j \in \{1, 2, \ldots, d\}$ and (46) follows.

4.3. An Empty ℓ_1-ball

Consider the following polytope:

$$P_d := \left\{ x \in \mathbb{R}^d : \sum_{j \in S} x_j + \sum_{j \notin S} (1 - x_j) \geqslant \frac{1}{2}\ \forall S \subseteq \{1, 2, \ldots, d\} \right\}. \tag{50}$$

Notice that any 0-1 vector x with $x_i = 1$ exactly for $i \in T$, does not satisfy the inequality corresponding to $S = \{1, 2, \ldots, d\} \backslash T$. Thus, $(P_d)_{\mathrm{I}} = \emptyset$. Denote by K_d the homogenization of P_d. Define, for $p \in \{1, 2, \ldots, d\}$,

$$L_{0p} : (x_0, x_1, \ldots x_{d-1})^{\mathrm{T}} \mapsto (x_0, x_1, \ldots, x_{p-1}, 0, x_p, \ldots, x_{d-1})^{\mathrm{T}},$$

$$L_{1p} : (x_0, x_1, \ldots x_{d-1})^{\mathrm{T}} \mapsto (x_0, x_1, \ldots, x_{p-1}, x_0, x_p, \ldots, x_{d-1})^{\mathrm{T}}, \tag{51}$$

$$Y_1 := \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \tag{52}$$

(53)

$$Y_d := \tfrac{1}{2}(L_{01} Y_{d-1} L_{01}^{\mathrm{T}} + L_{11} Y_{d-1} L_{11}^{\mathrm{T}}) \quad \text{for } d \in \{2, 3, \ldots\}. \tag{54}$$

Then, it is easily seen that Y_d defined in (54) coincides Y_d given below:

$$Y_d := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}. \tag{55}$$

For instance,

$$Y_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad L_{01} Y_1 L_{01}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad L_{11} Y_1 L_{11}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{56}$$

Hence,

$$Y_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{57}$$

Clearly, $v(d) = Y_d e_0$. Trivially, Y_1 satisfies (5), (9), and (10). Therefore, the same conditions are met by Y_d for $d \in \{2, 3, \ldots\}$. Thus, to prove $v(d) \in N_+^{d-1}(d)$, it remains to show that $Y_d e_j, Y_d f_j \in N_+^{d-2}(d)$ for $j \in \{1, 2, \ldots, d\}$. First, $v(1) \in N_+^0(1) = K_1$. Notice that for $j \in \{1, 2, \ldots, d\}$,

$$Y_d e_j = \tfrac{1}{2} L_{1j} v(d-1) \quad \text{and} \tag{58}$$

$$Y_d f_j = \tfrac{1}{2} L_{0j} v(d-1). \tag{59}$$

From the induction hypothesis, $v(d-1) \in N_+^{d-2}(d-1)$. But, for every $j \in \{1, 2, \ldots, d\}$, L_{i1} for $i \in \{0, 1\}$ are N_+-commutative and feasible. Thus, for $i \in \{0, 1\}$, $L_{i1} N_+^{d-2}(d-1) := L_{i1} N_+^{d-2}(K_{d-1}) \subseteq N_+^{d-2}(L_{i1} K_{d-1}) \subseteq N_+^{d-2}(K_d)$. Therefore, we have $Y_d e_j, Y_d f_j \in N_+^{d-2}(d)$. Hence, the proof.

Remark 15. Instead of (54), we can also use $(1/2d)\sum_{p=1}^{d} (L_{0p} Y_{d-1} L_{0p}^{\mathrm{T}} + L_{1p} Y_{d-1} L_{1p}^{\mathrm{T}})$ to obtain the same results.

4.4. TSP: ($\frac{4}{3}$)-conjecture related lower bounds

Since our techniques generalize those of Cook and Dash [10], their main results concerning N_+-rank of the TSP polytope fits into the framework.

Recently, Cheung [9] obtained, through an elegant analysis, some lower-bound results for the N_+-rank of various relaxations obtained from the subtour elimination polytope. Motivation of [9] is the so-called ($\frac{4}{3}$)-conjecture. The proof analyzes the eigenspaces of the individual matrices to prove that the claimed fractional vector indeed lies in $N_+^k(P)$. Our unifying approach is also directly applicable to this situation since the original proof uses only the well-established embeddings, all of which are M_+-commutative.

4.5. Packing, covering and set partition type problems

Very important and typical applications of lift-and-project methods have been in the general area of packing, covering and set partition type problems (see [26,4,29,2]). Lower-bound analyses for the results based on $N_\sharp$ operators can easily be covered by our unification. However, Sherali and Lee [29] work with Sherali–Adams reformulation–linearization technique (RLT). While operators like RLT, $N_\sharp$ and the one by Lasserre all belong to the same general lift-and-project family of operators, the lower-bound unification for RLT and Lasserre-type operators should be done in the language of “optimization over lattices interpretation” of the lift-and-project methods. Hence, these are not currently covered by our framework.

To extend our framework to RLT and Lasserre-type methods, one has to deal with two separate dimension increases in a single inductive step. One increase (same as what we had) is in the dimension of the original instance (e.g., we go from a k-clique to a $(k+2)$-clique). The second increase is in the order of the monomials used in obtaining the new higher-order relaxations.

5. Conclusion

We presented a unified proof technique for establishing lower bounds on the SDP-based lift-and-project rank of combinatorial optimization polyhedra. There are two obvious future research directions opened by our approach:

- Clearly, the lower bound established for a stronger operator directly applies to the weaker operator. However, to obtain better lower-bound results for the weaker operators, one needs to focus on those N_0- and N-commutative maps that are not N_+-commutative. In particular, complete characterization of N-commutative maps can be very useful in settling many open questions.

- It seems that our technique has a lot of potential for generalization to the so-called "optimization over lattices interpretation" of the lift-and-project methods (see [26,22]). Such a generalization would help analyze Sherali–Adams operator and Lasserre-type methods. Indeed, in the Appendix of [22], Laurent sketches "a tentative iterative proof" which has similarities to our framework.

Appendix A. Proof of Corollary 8

Assume that L is an automorphism of Q. Then, from (3) it follows that L^{T} is also an automorphism of Q^*. Hence, L^{T} preserves the set of extreme rays as well as the interior of Q^*. As e_0 is an interior ray, so is $L^{\mathrm{T}}e_0$. But, since $e_0 = e_i + f_i\ \forall\ i \in \{1, 2, \ldots, d\}$, there exist $\lambda_{i1}, \lambda_{i2} > 0$ such that

$$L^{\mathrm{T}}e_0 = \lambda_{i1}v_{i1} + \lambda_{i2}v_{i2} \quad \forall i \in \{1, 2, \ldots, d\},$$

where v_{ij} are distinct extreme rays $\{e_1, \ldots, e_d; f_1, \ldots, f_d\}$ of Q^*. Now, it is easy to see that for each i, v_{i1} and v_{i2} must be "complementary," that is, there exits j such that if $v_{i1} = e_j$ then $v_{i2} = f_j$ and vice versa. Furthermore, $\lambda_{i1} = \lambda_{i2}$ should hold. Hence, $L^{\mathrm{T}}e_0$ is a positive multiple of e_0. Also, we have shown that there are $\lambda > 0$ and a permutation $\sigma : \{1, 2, \ldots, d\} \to \{1, 2, \ldots, d\}$ such that

$$\{L^{\mathrm{T}}e_i, L^{\mathrm{T}}f_i\} = \{\lambda e_{\sigma(i)}, \lambda f_{\sigma(i)}\}. \tag{60}$$

The automorphism L of Q is a very special type of linear transformation. Since e_i's are d linearly independent vectors, L is unique up to the constant λ. In fact, (60) implies that L^{T} is of a very special form. For instance, suppose that $d = 3$, $\lambda = 1$, and L^{T} maps $e_1 \mapsto f_3$, $e_2 \mapsto e_1$, and $e_3 \mapsto f_2$. Then,

$$L^{\mathrm{T}} = \begin{bmatrix} e_0 & f_3 & e_1 & f_2 \end{bmatrix} = \left[\begin{array}{c|ccc} 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{array}\right]. \tag{61}$$

The generalization of (61) for general d is obvious. It says that an automorphism L^{T} is totally unimodular and a composite of permutation, flipping, and scaling. L^{T} in (61), for example, can be rewritten as

$$L^{\mathrm{T}} = \left[\begin{array}{c|ccc} 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]. \tag{62}$$

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